Expository Writing: Cosets of a Subgroup

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Here's some discussion of "obvious" weird stuff going on with cosets.

Suppose $H \leq G$. There's a natural bijection between left and right cosets, and it is

$$f_1(gH) = Hg^{-1}$$

This is somewhat strange, but we need this structure to ensure that this map is well-defined. Indeed if you take two representatives g_1H, g_2H, $g_1^{-1}g_2 \in H$. Hence $g_1^{-1}(g_2^{-1})^{-1} \in H$ and therefore $Hg_1^{-1} = Hg_2^{-1}$. The obvious map $f_2(gH) = Hg$ isn't necessarily well-defined. So f_1 performs "better" than f_2 here.

However, there's something strange here. If H is a Normal subgroup, then obvious map $f_2(gH) = Hg$ would be well-defined, and actually $gH = Hg$ so this obvious map f_2 is just the identity map. This map gives you the isomorphism between G/H and $H\backslash G$, while the natural bijection f_1 doesn't give you an isomorphism in general.

Something completely unrelated:

Proposition 1.1. *Suppose G is a finite group, $H \leq G$. Then exist $g_1, g_2, ..., g_k \in G$ such that the g_iH are all the left cosets and Hg_i are all the right cosets.*

Proof. Construct a bipartite graph, where the two vertex sets are the left cosets and right cosets, and the edges are the elements of G: $g \in G$ connects gH and Hg. Repeated edges don't matter.

This is a regular bipartite graph, so by Hall's Marriage Theorem there's a perfect matching. This edge set (of the perfect matching) is the representation we are trying to find. ∎