Expository Writing: Product Formula

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This article revolves around this famous formula:

Theorem 1.1 (Product Formula)**.** *If S, T are finite subgroups of G, then* $|ST||S \cap T| = |S||T|$.

Now we give a few interpretations of this theorem by putting it in a variety of contexts, which would give rise to different proofs of it.

1 Direct Proof

This is directly paraphrased from Rotman "An Introduction to the Theory of Groups".

Define a function $\phi : S \times T \longrightarrow ST$ by $(s,t) \mapsto st$. This is a surjective function, and so we claim that $\left|\phi^{-1}(x)\right| = |S \cap T|$. Then we show that $\phi^{-1}(st) = \{(sd, d^{-1}t) : d \in S \cap T\}$.

$\subseteq$ is clear, and it shouldn't be too complicated to show $\supseteq$. So we're done.

In this proof we clearly see what is going on: there are $|S \cap T|$ pairs (s,t) that maps to the same st by this weirdly defined equivalence relation $(sd, d^{-1}t)$.

2 Coset Correspondence/Second Isomorphism Theorem

Now given this formula, we can do some easy manipulations and see

$$\frac{|ST|}{|T|} = \frac{|S|}{|S \cap T|}$$

which seem to indicate some sort of correspondence between cosets of T represented by elements of ST (which is just a collection of all representations of some cosets of T), and cosets of $S \cap T$ in S. And indeed there is.

Consider the map $\varphi : S/S \cap T \longrightarrow ST/T$ by $s(S \cap T) \mapsto sT$.

1. This is a well defined map, as $S \cap T \subseteq T$.

2. This map is injective, as if $s_1T = s_2T$ for some $s_1, s_2 \in S$, then $s_1^{-1}s_2 \in T$ and hence $s_1^{-1}s_2 \in S \cap T$.

3. This map is surjective by definition of ST.

Therefore this is a natural one-to-one correspondence between these cosets, and the Product Formula follows immediately.

Furthermore, if $T \trianglelefteq G$ then ST is a group, $T \trianglelefteq ST$, $S \cap T \trianglelefteq S$, and this φ is actually an isomorphism. This is the Second Isomorphism Theorem.

3 "Double Action" and Double Coset

This last interpretation/context is inspired by the concept of "Double Coset" which arises occasionally in Exam/Practice Questions. The author was trying to extend the idea of Double Coset as a "Double Action" of two groups on a set, which then turns out to coincide with an already existing idea.

Definition 3.1 (Double Coset)**.** S, T subgroups of G. Then an $(S,T)-$double coset is a subset of G of the form SgT. These double cosets partition G. Notice they are not necessarily the same size then.

Now, the author noticed that if we interpret

1. Cosets as Orbit of the Action of a Subgroup on a group, then

2. Double Coset as "Orbit" of a "Double Action" of 2 Subgroups on the group.

So we define

Definition 3.2 ("Double Action")**.** S, T are groups. X a set. A double action of S and T on X are two actions $S \times X \longrightarrow X$, $T \times X \longrightarrow X$ such that $st(x) = ts(x)$ for any $s \in S, t \in T, x \in X$.

This easily corresponds to two group homomorphisms $\phi_S : S \longrightarrow Sym(X), \phi_T : T \longrightarrow Sym(X)$ such that $\phi_S(s)\phi_T(t) = \phi_T(t)\phi_S(s)$ for all $s \in S, t \in T$. This commuting property is added, so that "orbits" still is a partition of X, and the order S, T and T, S wouldn't matter. Giving it a bit of thought would have the reader realise this is actually a very natural and essential addition to the definition.

Now, the effect of $S, T \leq G$ on G by s, t acting on $g = sgt$ is such a double action. The orbits are the double cosets defined above.

However, we disappointingly discover that this is nothing but the same as an action of $S \times T$ on X: the 2 individual actions are the action of $S \times T$ restricted to $S \times \{e\}$ and $\{e\} \times T$ respectively, and they naturally extend to the action on $S \times T$ with the commuting property corresponding to the fact that

$$(s,e) \cdot (e,t) = (s,t) = (e,t) \cdot (s,e)$$

But this gives as another interpretation of the Product Formula.

Consider the action of $S \times T$ on G by $(s,t) \star g = sgt^{-1}$, or interpreted as a "double action". $ST = SeT$ is just the orbit of e under this action, and the stabiliser is just $\{(h,h) \mid h \in S \cap T\}$ with size $|S \cap T|$. By the orbit-stabilizer theorem,

$$|S||T| = |S \times T| = \left|\text{orb}(e)\right|\left|\text{stab}(e)\right| = |ST||S \cap T|$$