Expository Writing: Farey Sequences

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In this article, we look at Farey sequences, some nice properties of it, and some of its number theoretical implications.

Definition 1.1 (Farey Sequences)**.** The Farey Sequence F_n is the list of all fractions (in lowest terms) between 0 and 1 listed from small to big. For example,

$$F_7 = \frac{0}{1}, \frac{1}{7}, \frac{1}{6}, \frac{1}{5}, \frac{1}{4}, \frac{2}{7}, \frac{1}{3}, \frac{2}{5}, \frac{3}{7}, \frac{1}{2}, \frac{4}{7}, \frac{3}{5}, \frac{2}{3}, \frac{5}{7}, \frac{3}{4}, \frac{4}{5}, \frac{5}{6}, \frac{6}{7}, \frac{1}{1}.$$

We can consider the process of going from F_n to F_{n+1} to be the process of "inserting" fractions with denominator $n+1$ and numerator in $(\mathbb{Z}/(n+1)\mathbb{Z})^*$ into the sequence F_n. One may be able to observe some properties of this sequence immediately, and we will formally prove 2 of them later. Here's a small observation first.

Lemma 1.1. *If* $\frac{a}{b} < \frac{c}{d}$*, then* $\frac{a}{b} < \frac{a+c}{b+d} < \frac{c}{d}$*.*

Proof. As $\frac{a}{b} < \frac{c}{d}$, we have $bc - ad > 0$.

$$\frac{a+c}{b+d} = \frac{(a/b)(b+d) - ad/b + c}{b+d} = \frac{a}{b} + \frac{bc-ad}{b(b+d)} > \frac{a}{b}$$

$$\frac{a+c}{b+d} = \frac{a - bc/d + (c/d)(b+d)}{b+d} = \frac{c}{d} + \frac{ad-bc}{d(b+d)} < \frac{c}{d}$$

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Lemma 1.2. *If* $bc - ad = 1$*, and* $\frac{a}{b} < \frac{m}{n} < \frac{c}{d}$*, then* $n \geq b + d$*.*

Proof.

$$\frac{c}{d} - \frac{a}{b} = \frac{1}{bd}$$

Hence

$$\left(\frac{c}{d} - \frac{m}{n}\right) + \left(\frac{m}{n} - \frac{a}{b}\right) = \frac{1}{bd}$$

where both terms on the left are positive. Expand and multiply through we get

$$b(cn - dm) + d(bm - an) = n$$

and $LHS \geq b \cdot 1 + d \cdot 1 = b + d$. ■

Proposition 1.3. *If $\frac{a}{b} < \frac{c}{d}$ are adjacent terms in a Farey Sequence, then*
(1) $bc - ad = 1$.
(2) The first fraction inserted between these two numbers (that will occur in a later Farey Sequence) is $\frac{a+c}{b+d}$. In other words, the fraction with the least denominator that has size between $\frac{a}{b}$ and $\frac{c}{d}$ is $\frac{a+c}{b+d}$.

Remark. With (2), we can actually conclude that in (1), $bc - ad = 1$ if and only if $\frac{a}{b} < \frac{c}{d}$ are adjacent terms in some Farey Sequence.

Proof. This is a proof by induction. We let $P(n)$ be the following statement:

In the sequence F_n, all adjacent terms $\frac{a}{b} < \frac{c}{d}$ satisfy $bc - ad = 1$, and all terms in F_n with denominator n satisfies that its adjacent terms $\frac{a}{b} < \frac{m}{n} < \frac{c}{d}$ in F_n have $a + c = m, b + d = n$.

Easily, $P(n)$ holding for all n along with Lemma 1.1 proves Proposition 1.3.
We check base cases easily. Suppose $P(k)$ holds for all $k < n$, $n \geq 2$.
Then assume in F_n we gave adjacent terms $\frac{a}{b} < \frac{m}{n} < \frac{c}{d}$, then $\frac{a}{b} < \frac{c}{d}$ are adjacent terms in F_{n-1} (need elaboration, but should be clear that $b, d \neq n$ so these are both in F_{n-1}). Hence $bc - ad = 1$ by $P(n-1)$. By Lemma 1.2, we have $b + d \leq n$. However if $b + d < n$, then $\frac{a}{b} < \frac{a+c}{b+d} < \frac{c}{d}$ and $\frac{a+c}{b+d}$ should be in F_n, contradicting our assumption about adjacent terms.
Therefore $b + d = n$. By Lemma 1.1 it's clear that therefore $a + c = m$.
Now we know where the new inserted terms go in F_n and what they are, it's easy to verify that $bc - ad = 1$ still holds for all adjacent terms in F_n. By principle of mathematical induction, we are done.

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Now we prove a nontrivial result regarding how good rational approximation can be with given "budget" (size of denominator) using the above.

Theorem 1.4. *Suppose α is irrational. Then there are infinitely many $p, q \in \mathbb{Z}$ such that*

$$\left|\alpha - \frac{p}{q}\right| < \frac{1}{q^2}$$

Proof. It's easy to see we only need to prove for $\alpha \in (0, 1)$.
For any Farey Sequence F_n, suppose $\frac{a}{b} < \frac{c}{d}$ are adjacent terms and $\frac{a}{b} < \alpha < \frac{c}{d}$. Then

$$\left|\alpha - \frac{a}{b}\right|, \left|\alpha - \frac{c}{d}\right| < \frac{c}{d} - \frac{a}{b} = \frac{1}{bd} < \frac{1}{\min(b, d)^2}$$

by properties of the Farey Sequence. Hence either $\frac{a}{b}$ or $\frac{c}{d}$ would satisfy the condition in the theorem.
Now it should be obvious to see that taking all Farey Sequences, while some may give us the same rational approximation, would generate an infinite list of such fractions. ■